PROGRESS ON CROSSING NUMBER PROBLEMS

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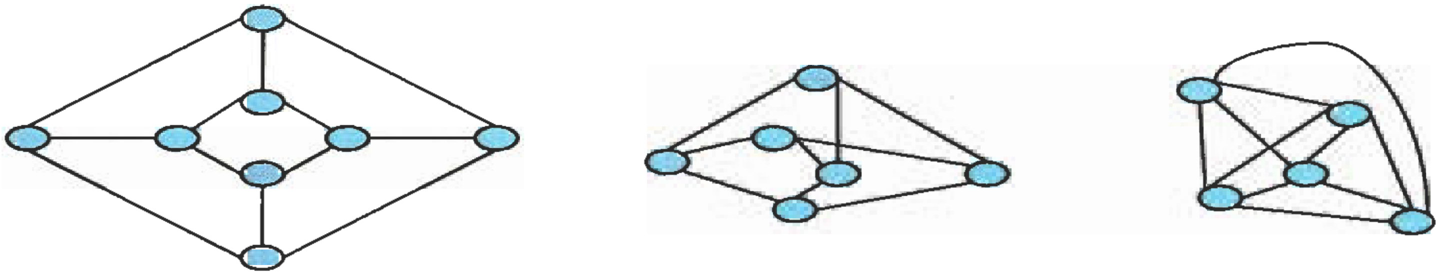
> “There were some kilns where the bricks were made and some open storage yards where the bricks were stored. All the kilns were connected by rail with all storage yards. … the trouble was only at crossings. The trucks generally jumped the rails there, and the bricks fell out of them; in short this caused a lot of trouble and loss of time … the idea occurred to me that this loss of time could have been minimized if the number of crossings of the rails had been minimized. But what is the minimum number of crossings?”
>
> P. Turán

Planar graphs:

- Vertices ~ points in the plane
- Edges ~ curves connecting the two endpoints of the edge
- Edges do not cross each other
- Edges do not go through other points of the graph than their endpoints.



Crossing number: cr(G) (Turán, 1942)

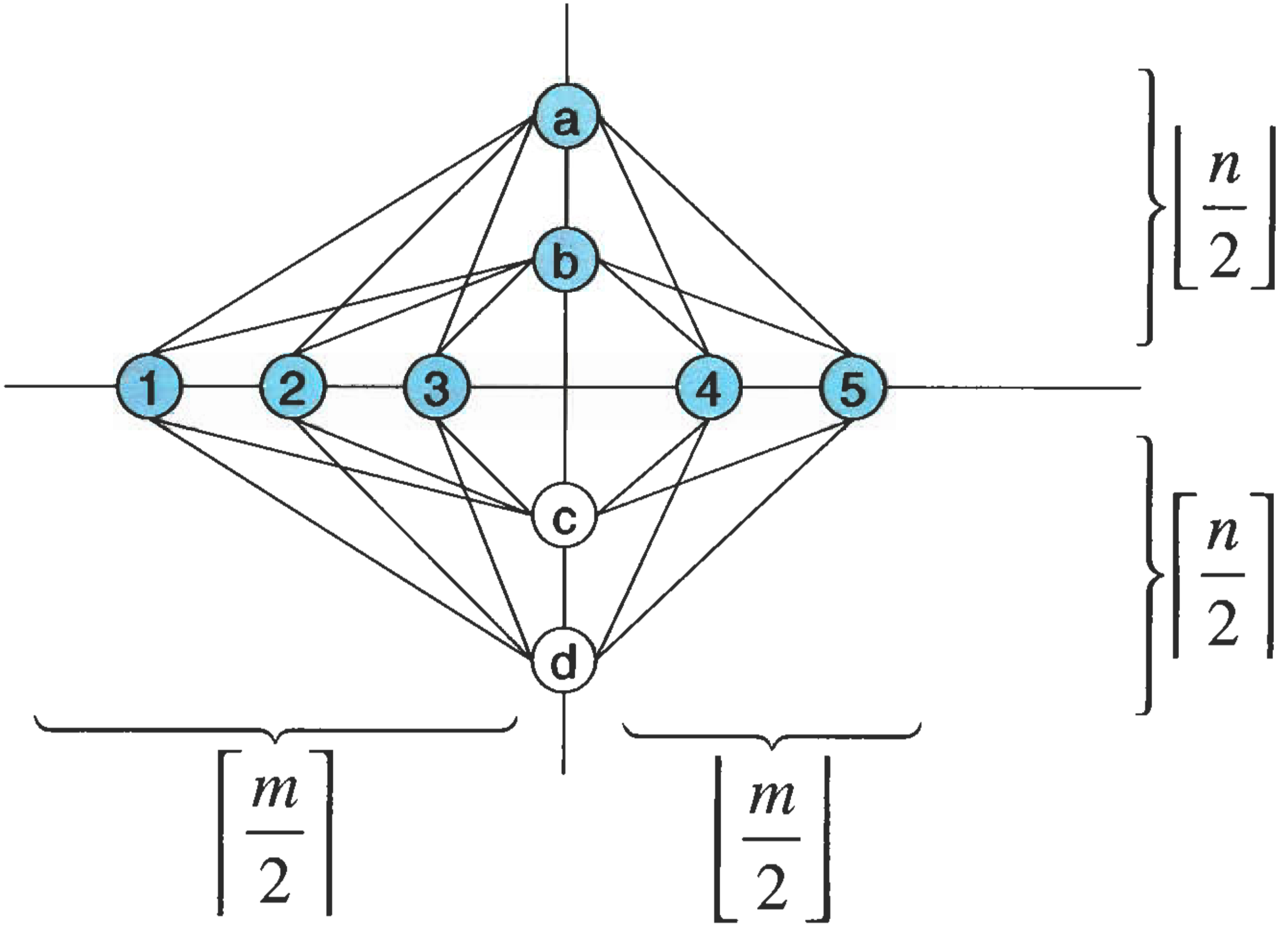
The smallest possible number of edge-crossings in a drawing of the graph G in the plane.

G is planar $\leftrightarrow$ cr(G) =0

Zarankiewicz conjecture:

$$\operatorname{cr}\left(K_{n,m}\right)=\left\lfloor\frac{n}{2}\right\rfloor\cdot\left\lfloor\frac{n-1}{2}\right\rfloor\cdot\left\lfloor\frac{m}{2}\right\rfloor\cdot\left\lfloor\frac{m-1}{2}\right\rfloor$$

Zarankiewicz drawing (for $K_{4,5}$):



Theorem (Fáry 1948) G has planar drawing in straight line segments iff G is planar.

Question: Can the crossing number be always realized with straight line drawing? $\mathrm{cr}(G) \le \text{CR-LIN}(G)$

NO! (Guy 1972) $\mathrm{cr}(K_9) < \text{CR-LIN}(K_9)$

Bienstock-Dean 1992 There exists G with $\mathrm{cr}(G) = 4$ and CR-LIN(G) arbitrarily large .

Zarankiewicz 1954

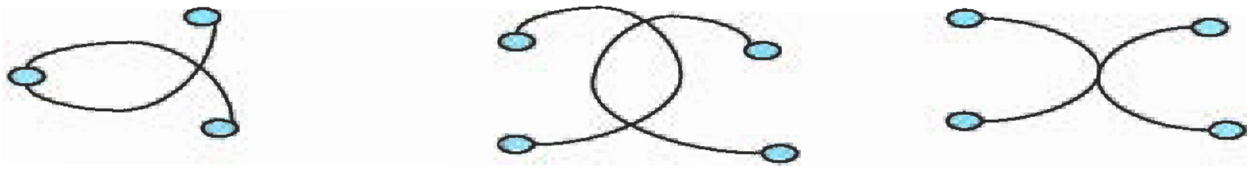
Urbanik 1955

Kainen and Ringel GAP!

Guy 1969 “The decline and fall of Zarankiewicz’s theorem”

- True for $m \leq 6$ (Kleitman 1970)
- Smallest counterexample is $K_{\text{odd,odd}}$.
- True for $K_{7,7}$ and $K_{7,9}$
- Open for $K_{7,11}$ and $K_{9,9}$
- $\lim_{n\to\infty} \frac{\operatorname{cr}\left(K_{n,n}\right)}{n^4}$ exists (by counting method); = ??

cr(G) does not change if the following are forbidden:



Pach-Tóth 1998 “Which crossing number is it anyway?” (Mohar 1995)

CR-PAIR(G) = min # crossing pairs of edges

$$\text{CR-PAIR}(G) \leq \text{cr}(G)$$

Imre Lakatos “Proofs and refutations”

CR-ODD(G) = min # odd times crossing pairs of edges (Tutte 1970)

CR-IODD(G) = (only non-adjacent edge pairs count)

Chojnacki 1934 CR-IODD(K_5) = 1, CR-IODD($K_{3,3}$) = 1,

$$\text{CR-IODD}(G) \leq \text{CR-ODD}(G) \leq \text{CR-PAIR}(G) \leq \text{cr}(G)$$

PROBLEM: Are they always equal?

Computational complexity

$\mathrm{cr}(G) \leq k$ NP-complete Garey-Johnson 1983

$$\left.\begin{array}{ll} \text{CR-PAIR}(G) \leq k & \text{NP-complete} \\ \text{CR-ODD}(G) \leq k & \text{NP-complete} \end{array}\right\} \text{Pach-Tóth 1998}$$

Leighton-Rao 1988 $\log^4 n$ times approximation algorithm to $n + \mathrm{cr}(G)$

Even-Guha-Schieber 2000 $\log^3 n$ times approximation algorithm to $n + \mathrm{cr}(G)$

GAREY-JOHNSON 1983

"$\mathrm{cr}(G) \le k$" IS NP-COMPLETE

PACH-TÓTH

FOR OTHER CROSSING NUMBERS

UNLIKELY

"$\mathrm{cr}(G) \le k$" $\in$ CO-NP

IS THERE AN OPTIMALITY CRITERION THAT *SOMETIMES* CAN BE CHECKED?

$Z_{n,m}$ = Zarankiewicz drawing of $K_{n,m}$

For $e=\{x,y\}\in E\left(K_{n,m}\right)$, let $A_e \subseteq V\left(K_{n,m}\right)-\{x,y\}$ arbitrary

$e,f \in E\left(K_{n,m}\right)$, $e\cap f=\varnothing$ ODD pair: $\left|f\cap A_e\right|+\left|e\cap A_f\right|$ is odd

$(e,f)=$ random pair of edges with $e\cap f=\varnothing$

Conjecture:

$$\mathbb{P}\left(\left\{(e,f)\text{ not crosses in }Z_{n,m}\right\}\wedge\left\{(e,f)\text{ is ODD}\right\}\right)\geq$$
$$\mathbb{P}\left(\left\{(e,f)\text{ crosses in }Z_{n,m}\right\}\wedge\left\{(e,f)\text{ is ODD}\right\}\right)$$

(This would imply the Zarankiewicz conjecture, but not known to be equivalent with it)

Methods

- Euler's Formula: $e > 3n - 6 \quad \longrightarrow \quad \mathrm{cr}(G) \geq 1$

  iteration yields: $\mathrm{cr}(G) \geq e - 3n + 6$

  $$\mathrm{cr}(G) \geq e - \frac{g}{g-2} \cdot (n-2),$$ where $g =$ girth of G

- Theorem (Ajtai, Chvátal, Newborn, Szemerédi 1982; Leighton 1983):

  $$\mathrm{cr}(G) \geq \frac{1}{64} \cdot \frac{e^3}{n^2} \quad \text{or} \quad e \leq 4n \quad .$$

- bisection width
- counting method
- graph embedding

Bisection width lower bound

$$E(A,B) = \#\text{edges betwen } A,B$$

$$b(G) = \min_{\substack{A \cup B = V \\ |A|,|B| \geq \frac{n}{3}}} E(A,B)$$

Theorem (Leighton 1982; Sykora-Vrto 1993; Pach-Shahroki-Szegedi 1994):

$$b(G) \leq 10\sqrt{\mathrm{cr}(G)} + 2\sqrt{\sum_{i=1}^{n} d_i^2}$$

BISECTION WIDTH:

PACH-SHAHROKHI-SZEGEDY [1994]

$$(1.58)^2\left(16\,cr(G)+\sum_{i\in V} d_i^2\right) \geq b(G)^2$$

WHERE $d_i = i^{th}$ DEGREE

$$b(G) = \min_{|V_1|,|V_2|\geq n/3} E(V_1,V_2)$$

PROOF TO A SPECIAL CASE (LEIGHTON)

G DEGREE BOUNDED

$$CR(G)+n = \Omega(b^2(G))$$

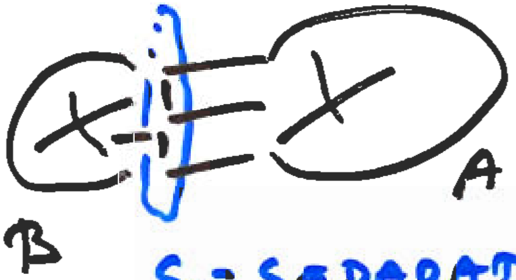
LIPTON-TARJAN SEPARATOR THEOREM:

G n-VERTEX PLANAR GRAPH

$w \geq 0$ WEIGHT ON $V(G)$, $\sum_v w(v) = 1$

THEN $V(G)$ CAN BE PARTITIONED INTO A, S, B:

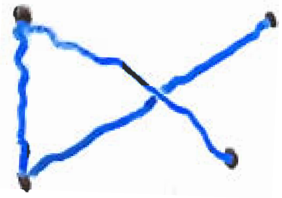
- NO EDGES BETWEEN A, B
- $w(A) \leq 2/3$, $w(B) \leq 2/3$
- $|S| \leq \sqrt{8n}$

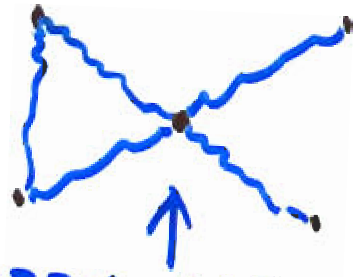


S = SEPARATOR

PROOF:

DRAWING OF G $\longrightarrow$ G'



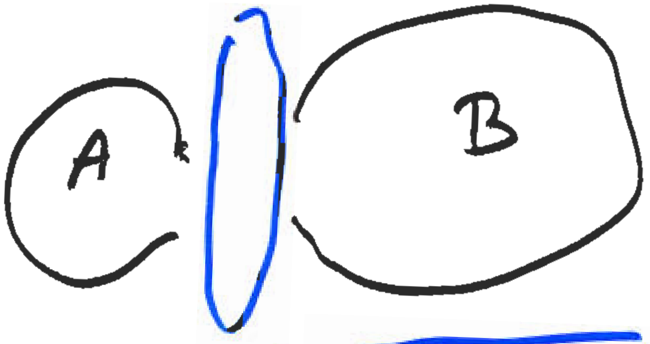


NEW VERTEX ADDED
AT CROSSINGS

G' PLANAR

G' has $n + cr(G)$ vertices

APPLY LIPTON-TARJAN TO G'
WITH $w(\text{OLD}) = \frac{1}{n}$ $w(\text{NEW}) = 0$



$w(A) \leq \frac{2}{3}$

$w(B) \leq \frac{2}{3}$

$|S| \leq \sqrt{8(cr(G)+n)}$

$b(G) \leq$ # EDGES INCIDENT TO S

SPLIT S PROPERTY

GRAPH EMBEDDING:

$G_1=(V_1,E_1)$ $G_2=(V_2,E_2)$ $|V_1|\le|V_2|$

EMBEDDING $\omega : G_1 \to G_2$
IS PAIR OF INJECTIONS (ϕ,ψ)

$\phi : V_1 \to V_2$

$\psi : E_1 \to$ PATH SET OF G_2

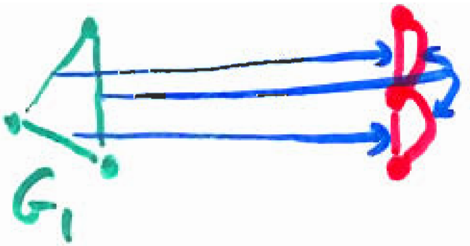
$uv \in E_1 \Longrightarrow \psi(uv)$ IS $\phi(u)\phi(v)$ PATH

$$\mu_\omega(e)=\{f\in E_2 : e\in\psi(f)\}$$

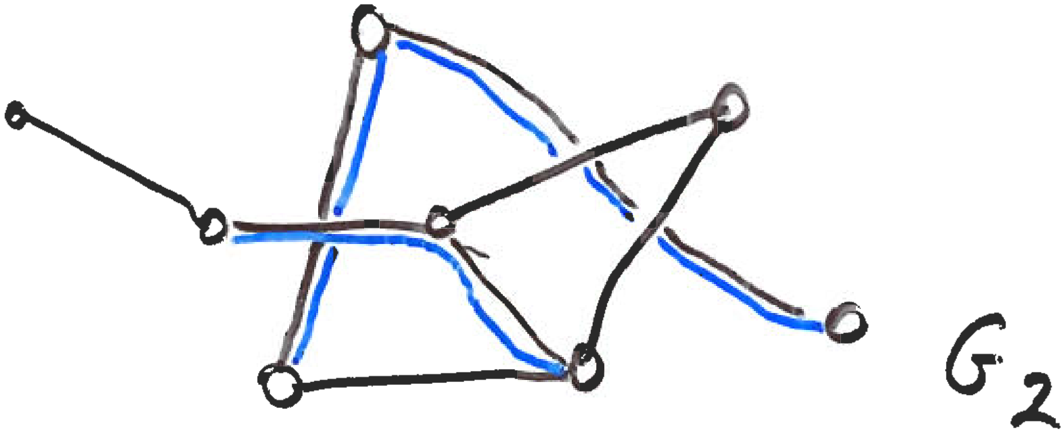
$$m_\omega(u)=\{f\in E_2 : u\in\psi(f)\}$$

$$\mu_\omega=\max_{e\in E_2}\mu_\omega(e)$$

$$m_\omega=\max_{u\in V_2} m_\omega(u)$$



DRAWING OF G_2 $\rightsquigarrow$ DRAWING OF G_1



BLUE CROSSINGS AT G_1

BLACK CROSSINGS	VERTICES
$\leq \mu_w^2$ per black crossing	$\leq \binom{m_w}{2}$ per vertices

$$cr(G_1) \leq \mu_w^2 \cdot cr(G_2) + \frac{n}{2} m_w^2$$

THEOREM

SHAHROKHI S. S. V.

$$cr(G_2) \geq \frac{cr(G_1)}{\mu_\omega^2} - \frac{n}{2}\left(\frac{m_\omega}{\mu_\omega}\right)^2$$

$n = |V_2|$

YIELDS

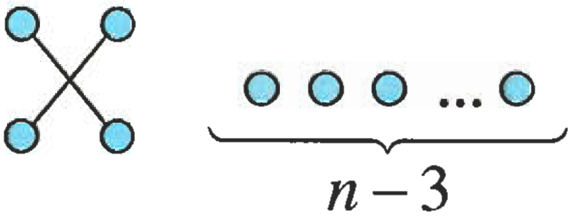
$$cr(Q_n) \geq c \cdot 4^n$$

(TIGHT WITHIN CONSTANT MULTIDLICATIV FACTOR)

Counting method

Count crossings in copies of K_n in a drawn K_{n+1}

$$\mathrm{cr}(K_{n+1}) \geq \frac{(n+1)\mathrm{cr}(K_n)}{n-3}$$



$$\frac{\mathrm{cr}(K_{n+1})}{\binom{n+1}{4}} \geq \frac{\mathrm{cr}(K_n)}{\binom{n}{4}}$$

$$\exists \lim_{n\to\infty} \frac{\mathrm{cr}(K_n)}{\binom{n}{4}} \qquad \mathrm{cr}(K_n) \sim cn^4 \qquad \frac{1}{80} \leq c \leq \frac{1}{64}$$

CONJECTURE: $\mathrm{cr}(K_n) = \frac{1}{4} \cdot \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n-1}{2} \right\rfloor \cdot \left\lfloor \frac{n-2}{2} \right\rfloor \cdot \left\lfloor \frac{n-3}{2} \right\rfloor$

ÁBREGO, FERNÁNDEZ-MERCHANT
2003

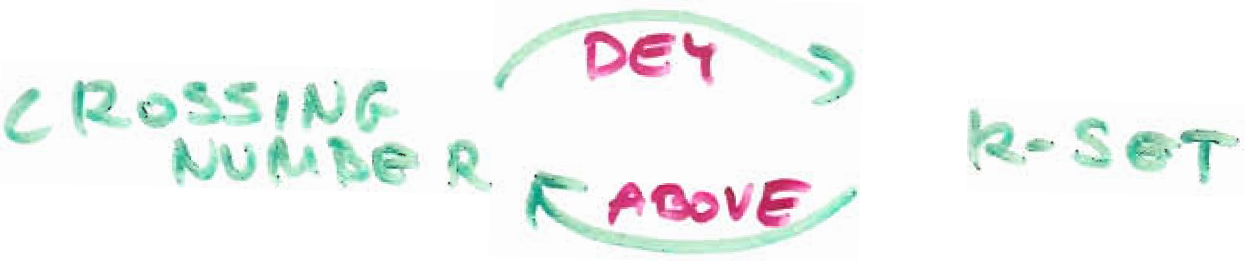
$CR\text{-}LIN(K_n) \gtrsim \frac{n^4}{64}$

LOVÁSZ, VESZTERGOMBI, WAGNER,
WELZL 2003

$CR\text{-}LIN(K_n) \geq \left(\frac{1}{64}+\varepsilon\right)n^4$

BALOGH-SALAZAR 2004

BETTER ε



UPPER BOUNDS FOR $CR\text{-}LIN(K_n)$:

AICHHOLZER, AURENHAMMER, KRASSER

[EXACT UP TO K_{16}]

DE KLERK, MAHARRY, PASECHNIK, SALAZAR

RICHTER 2004

83% OF

ZARANKIEWICZ

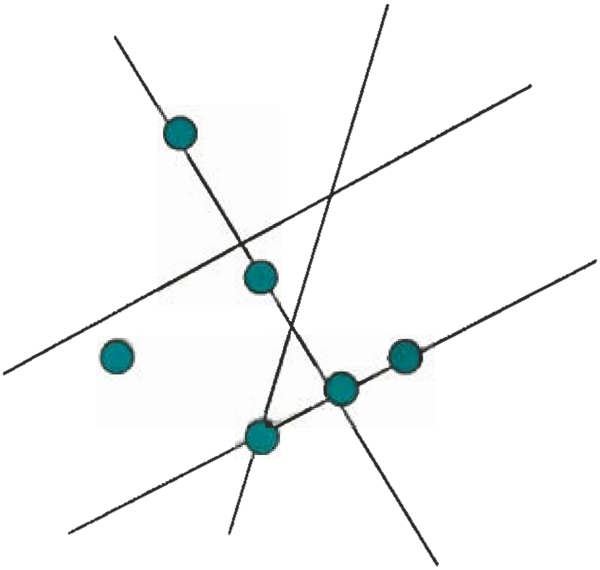
CONJECTURE

Why do we bother with a graph parameter that we can not evaluate for complete graphs?

- VLSI
- Mathematical applications

Erdős-Purdy conjecture (1970's): n points m lines in the plane have at most $c\left(n+m+(nm)^{2/3}\right)$ incidences

The result is tight up to a constant multiplicative factor.

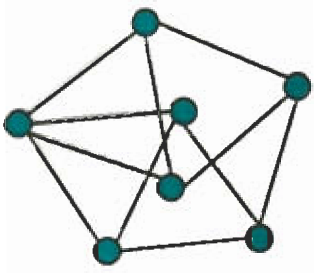


#incidences = 7

Szemerédi-Trotter Theorem (1982):

$c=10^{60}$ makes it true.

Erdős' unit distance conjecture (1946): For all $\varepsilon > 0$, the number of unit distances among n points in the plane is at most $c_\varepsilon n^{1+\varepsilon}$.



$n=7$

11 unit distances

Erdős' construction (1946): The number of unit distances can be $n^{1+c/\log\log n}$ ($> n\log n$, even $n\log^k n$)

$145=12^2+1^2=9^2+8^2$

Erdős' distinct distances conjecture (1946): The number of distinct distances among n points in the plane $> cn/\sqrt{\log n}$ (he had a matching construction based on the square lattice)

My crossing number method to prove inequalities in combinatorial geometry (1997):

- Consider an “appropriate” graph drawn in the plane.
- Set lower and upper bounds for its crossing number.
- Analyze the resulting inequality.

Some applications:

- My short proof of the Szemerédi-Trotter theorem.
- My short proof for the best result on the unit distance problem.
- Progress on the distinct distances problem.

FURTHER APPLICATIONS OF CROSSING NUMBER METHOD AND/OR SZEMERÉDI-TROTTER THEOREMS

ELEKES n REALS HAVE $n^{5/4}$ DISTINCT SUMS OR PRODUCTS

PACH-SHARIR INCIDENCES OF POINTS, TRANSLATES OF STRICTLY CONVEX CLOSED CURVES

ANDREWS (IOSEVICH PROOF) CONVEX POLYGON WITH n LATTICE VERTICES $n = O(\text{Area}^{1/3})$

FUGLEDE (IOSEVICH et al.) $D = \{|x| \leq r\} \subseteq \mathbb{R}^d$ NOT SPECTRAL FOR $d \geq 2$

DEY # OF PLANAR k-sets $\leq 7n(k+2)^{1/3}$

# of unit distances	# of distinct distances
$< cn^{3/2}$ (Erdős 1946)	$> cn^{1/2}$ (Erdős 1946)
$< o(n^{3/2})$ (Józsa, Szemerédi 1973)	$> cn^{2/3}$ (Moser 1952)
$< n^{1.444\ldots}$ (Beck, Spencer 1984)	$> cn^{5/7}$ (Fan Chung 1984)
$< cn^{4/3}$ (Spencer, Szemerédi, Trotter 1984)	$> cn^{58/81-\varepsilon}$ (Beck 1984)
	$> n^{4/5}/\log^{c} n$ (Fan Chung, Szemerédi, Trotter 1992)
	$> cn^{3/4}$ from a single point (Clarkson, Edelsbrunner, Guibas, Sharir, Welzl 1990)
	$> cn^{4/5}$ from a single point (Szekely 1997)
	$> cn^{6/7}$ (Solymosi, Tóth 2001)
	$> cn^{4e/(5e-1)-o(1)}$ (Tardos 2002)

$> cn^{19/22-o(1)}$ (KATZ 2003)

$> cn^{\frac{42-14e}{55-16e}-o(1)}$ (KATZ-TARDOS 2004)

Leighton theorem $\operatorname{cr}(G) \geq \frac{1}{64} \cdot \frac{e^3}{n^2}$ or $e \leq 4n$.

Proof (folklore): D = drawing of G realizing cr(G).

Pick vertices of D independently with probability $p = 4n/e$.

Picked vertex set induces subgraph G' with subdrawing D' (with n' points and e' edges, which are random variables).

For every outcome:

# of crossings in D' $\geq e' - 3n'$

Taking expectation and applying linearity gives

$$p^4 \operatorname{cr}(G) \geq p^2 e - 3pn,$$

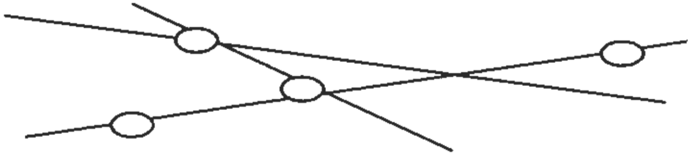
i.e. $\operatorname{cr}(G) \geq e/p^2 - 3n/p^3 = e^3/(64n^2)$.

Szemerédi-Trotter Theorem (1982): n points m lines in the plane have at most $c\left(n+m+(nm)^{2/3}\right)$ incidences.

Proof (Szekely 1997): Wlog there is at least one point on every line.

Construct a graph G drawn in the plane:

- vertices of the graph G = the n points.
- edges of the graph G = line segments connecting consecutive points on the m lines.
- # incidences on a line = # of edges on that line + 1.
- e = #incidences – m.



$$\binom{m}{2} \geq \mathrm{cr}(G) \geq \frac{1}{64} \cdot \frac{(\#\text{incidences} - m)^3}{n^2} \text{ or } \#\text{incidences} - m \leq 4n.$$

ERDŐS-SZEMERÉDI PROBLEM:

$A \subseteq \mathbb{R}$

HOW SMALL CAN BE $g(n) = \max_{|A|=n} \{|A+A|, |A \cdot A|\}$?

ERDŐS-SZEMERÉDI, 1983 $g(n) > n^{1+\varepsilon}$

NATHANSON $g(n) > n^{32/31}$

FORD $g(n) > n^{16/15}$

ELEKES, 1997 $g(n) > n^{5/4}$

PROOF (ELEKES)

$P = (A \cdot A) \times (A + A) = \{(a \cdot b, c + d) : a, b, c, d \in A\}$

$L = \{ \{(ax, a' + x) : x \in \mathbb{R}\} : a, a' \in A\}$

$$n^3 \leq \#\text{incidences} = O\left(|P|^{2/3} |L|^{2/3} + |P| + |L|\right)$$

$$|L| = n^2$$

$$n^{5/2} \leq |P| \leq g^2(n)$$